where ρ_i is the radius of curvature of the apex of the i-th inclusion.

As an illustration, we present the value of a numerical analysis of (2.13) for the case of an infinite space with two identical spheroidal inclusions. A field of uniaxial tension in the direction of the axis of the inclusions is given at infinity. The results are represented in Fig. 2 for different values of the parameter $\beta = a/c$ where $\nu = \nu_1 = 0.3$, and $H/a = 2$.

Also presented for comparison are the data from /6/ (the dashed line) obtained by the method of equivalent inclusion. As is seen from the curves, there is good agreement between the results even for fairly thick inclusions and a broad range of variation of the parameter $\varepsilon_1 = E_1/E$.

The results in /7-9/ follow from (2.13) and (2.16) as special cases.

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THE STANDARD EQUATION METHOD IN THE DYNAMICS OF STRUCTURALLY INHOMOGENEOUS ELASTIC MEDIA*

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The development of the standard equation method is examined for studying harmonic wave propagation in stochastically inhomogeneous elastic media. The Helmholtz operator equation describing the propagation of a mean scalar field in a medium is investigated as the standard equation. For an arbitrary correlation function of the elastic coefficients of the medium, the roots of the dispersion equation are found by expanding them in a series in the dispersion parameter, and the eigenvectors of the operator are correspondingly determined approximately. For media of the exponential class, the roots and eigenfunctions of the standard problem are determined exactly. Results obtained in solving the standard problem, are used in investigating wave propagation in elastic media; the roots and eigenvectors are found in the form of a series expansion in the dispersion. A relationship is set up between the spectra of the elastic operator and the operator of the standard problem. Formulas are obtained to find the mean elastic fields (including the eigenvectors) in terms of the mean standard functions in the form of scattering series.

The elastic operator in an isotropic homogeneous body has eigenvectors in the form of longitudinal and transverse waves satisfying the Helmholtz equations. The eigenvalues and vectors of an elastic operator are a set of eigenvalues and vectors of the Helmholtz operator /1/. The elasticity equations do not split into Helmholtz equations or scalar equations in the general case in an inhomogeneous medium. This can be done for high

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frequencies /2/ and certain particular kinds of inhomogeneities /3/.

The method of standard equations /4,5/ enables the form in which the eigenfunctions and eigenvalues of the problem under investigation must be sought, to be determined for stochastically inhomogeneous media, the equations for the mean field are integro-differential: the medium possesses spatial dispersion. The dispersion equations are transcendental and the roots can only be found approximately. We take the Helmholtz operator equation /6/ as the standard equation, then the roots of the dispersion equation and the eigenfunctions are found exactly for the class of media characterized by an exponential correlation function. The results of solving the standard problem are used in solving the elastic problem, where the eigenvalues of the elastic operator are expressed in terms of the roots and eigenvectors of the standard problem. The standard solution discribes qualitatively wave propagation in a structurally inhomogeneous medium, which enables us to speak of the similarity between corresponding dispersion laws and damping /7/. The dimensionless parameters in which the quantities in both problems are expanded are the dispersion and the product of the correlation radius by the wave number /7,8/.

1. The displacement vector $\mathbf{u}$ in a harmonic wave being propagating in a stochastically inhomogeneous elastic medium satisfies the equations

$$L\mathbf{u} + \rho_0\omega^2\mathbf{u} = 0, \quad L_{ik} = \nabla_j\lambda_{ijkl}(\mathbf{x})\nabla_l \tag{1.1}$$

Here λ_{ijkl} depends on $\mathbf{x}$ in a random manner, ρ_0 is the constant density, and ω is the frequency.

Introducing the effective elastic operator Λ^* by the relationship

$$\langle\lambda_{ijkl}u_{k,l}\rangle = \Lambda_{ijkl}^*\langle u_{k,l}\rangle \quad u_{k,l} = \frac{\partial u_k}{\partial x_l} \tag{1.2}$$

and taking the average of (1.1), we obtain a closed system of equations for $\langle\mathbf{u}\rangle$

$$L^*\langle\mathbf{u}\rangle + \rho_0\omega^2\langle\mathbf{u}\rangle = 0, \quad L_{ik}^* = \nabla_j\Lambda_{ijkl}^*\nabla_l \tag{1.3}$$

For a statistically isotropic homogeneous medium, the elastic operator Λ^* has the form

$$\Lambda^* = \int\lambda^*(\mathbf{x} - \mathbf{x}_1)\,d\mathbf{x}_1 \tag{1.4}$$

The eigenvectors of the operator (1.4) are on the average plane waves /6/

$$\langle\mathbf{u}(\mathbf{x})\rangle = \mathbf{u}(\mathbf{q})\,e^{i\mathbf{q}\cdot\mathbf{x}} \tag{1.5}$$

Longitudinal $\langle\mathbf{u}^l\rangle$ and transverse $\langle\mathbf{u}^t\rangle$ waves exist in the medium under consideration, and we have

$$\langle\mathbf{u}\rangle = \langle\mathbf{u}^l\rangle + \langle\mathbf{u}^t\rangle, \quad \langle\mathbf{u}^\alpha\rangle = \mathbf{u}^\alpha(\mathbf{q}_\alpha)\,e^{i\mathbf{q}_\alpha\cdot\mathbf{x}} \tag{1.6}$$

Taking account of (1.5) and (1.6), dispersion equations in q_α follow from equations (1.3)

$$\Delta_\alpha = q_\alpha^2 - \rho_0\omega^2\Lambda_\alpha^{-1}(q_\alpha, \omega) = 0 \tag{1.7}$$

Here Λ_α are the eigenvalues of the tensor

$$L_{ik}^*(\mathbf{q}) = \int L_{ik}^*(\mathbf{r})\,e^{-i\mathbf{q}\cdot\mathbf{r}}\,d\mathbf{r} \tag{1.8}$$

The explicit analytic form $\Lambda_\alpha(q_\alpha, \omega)$ depends on the specific form of the correlation dependence $\lambda_{ijkl}(\mathbf{x})$. However, independently of the form of the correlation function for a statistically isotropic homogeneous medium, the quantity Λ_α depends on q_α^2 and can be represented in the form of an entire function of q_α^2 in the complex q_α^2 plane.

To find $\Lambda_\alpha(q_\alpha^2, \omega)$ taking multiple scattering into account, we apply the method of replacement of the field quantities /6/; we then obtain

$$\begin{aligned} &\Lambda_l(q_l, \omega) = \Lambda_{l_0} + \Gamma^*(1 - L^{(l)}\Gamma^*)^{-1} + {}^4/_3\Lambda_t(q_l, \omega) \\ &\Lambda_t(q_t, \omega) = \Lambda_{t_0} + \Gamma_2^*(1 - L^{(t)}\Gamma_2^*)^{-1} \\ &\Lambda_{t0} = \mu_0, \;\; \Lambda_{l0} = K_0 + {}^4/_3\mu_0 \end{aligned} \tag{1.9}$$

Here K_0, μ_0 are the effective bulk and shear elastic moduli, $\Gamma^*(q, \omega)$, $\Gamma_2^*(q, \omega)$ are eigenvalues of the polarizability operator Γ^* of the medium under consideration

$$\begin{aligned} &\Gamma_{ijkl}^* = \int\Gamma_{ijkl}^*(\mathbf{x} - \mathbf{x}_1)\,d\mathbf{x}_1, \quad \mathbf{r} = \mathbf{x} - \mathbf{x}_1 \\ &\Gamma_{ijkl}^*(\mathbf{q}, \omega) = R_{ijst}^{mnkl}\gamma_{smnt}^*(\mathbf{q}, \omega) \\ &\gamma^*(\mathbf{q}, \omega) = \int\gamma^*(\mathbf{r}, \omega)\,e^{-i\mathbf{q}\cdot\mathbf{r}}\,d\mathbf{r}, \quad \Gamma_{nmst}^*(\mathbf{r}) = R_{mnql}^{past}\gamma_{qpal}^*(\mathbf{r}) \end{aligned} \tag{1.10}$$

$$\gamma^*_{qp\alpha l}(\mathbf{r}) = R(\mathbf{r})\, G^{(R)}_{qp,\,\alpha l}(\mathbf{r}), \quad R^{p\alpha st}_{nmql}(\mathbf{r}) = R(\mathbf{r})\, R^{p\alpha st}_{nmql}$$

Here $R^{p\alpha st}_{nmql}$ is a constant tensor governing the tensor dependence of the correlation tensor $R^{p\alpha st}_{nmql}(\mathbf{r})$ of the field $\dot{\gamma}_{ijkl}(\mathbf{x})$: $R^{p\alpha st}_{nmql}(\mathbf{r}) = \langle \gamma_{p\alpha st}(\mathbf{x})\, \gamma_{nmql}(\mathbf{x}_1)\rangle$, and the correlation function $R(r)$ governs the coordinate dependence of the correlation tensor $R^{p\alpha st}_{nmql}(\mathbf{r})$. For a strongly isotropic medium, the tensors $\Gamma^*_{ijkl}(\mathbf{q}, \omega)$, $\lambda^*_{ijkl}(\mathbf{q}, \omega)$, $\gamma^*_{ijkl}(\mathbf{q}, \omega)$ have the form

$$\begin{aligned} &F^*(q, \omega) = F_1^*(q, \omega) I^2 + F_2^*(q, \omega) I^1 \quad (F_i^* = \Gamma_i^*, \lambda_i^*, \gamma_i^*) \\ &I^2 = \delta_{ij}\delta_{kl}, \quad I^1 = {}^1/_2 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\Gamma^* = \Gamma_1^* + {}^2/_3\Gamma_2^*, \quad K^* = \lambda_1^* + {}^2/_3\mu^*, \quad \gamma^* = \gamma_1^* + {}^2/_3\gamma_2^* \\ &\Gamma_1^* = 3(3\gamma_1^* + 4\gamma_2^*), \quad \Gamma_2^* = 3(2\gamma_1^* + \gamma_2^*), \quad F_i^* = 0 \ (i = 3, 4, 5, 6) \end{aligned} \tag{1.11}$$

Without specifying the form of the correlation function, we calculate the components of $\gamma^*_{ijkl}(q, \omega)$ by means of the third formula in (1.10). We now consider the spherical coordinate system under the integral and integrate with respect to the angles, we then obtain /6,9/

$$\begin{aligned} &\gamma_1^* = \theta_t [P^2_{2(t)} - iP^2_{2(t)} - P^2_{1(t)}\varkappa_t] + 2^{-1}\gamma_2^* \\ &\gamma_2^* = -2\theta_\alpha [P^3_{4(\alpha)} - 3iP^4_{4(\alpha)} + 2P^4_{3(\alpha)} + 2iP^3_{3(\alpha)}\varkappa_\alpha + P^3_{2(\alpha)}\varkappa_\alpha^2]^l_t \\ &P^p_{n(\alpha)} = \int_0^\infty j_n(z)\, y_\alpha^{-p} e^{iy_\alpha} R(r)\, r^2\, dr, \quad \alpha = l, t \\ &z = qr, \quad y_\alpha = k_\alpha r, \quad \theta_\alpha = k_\alpha^3 (\rho_0 c_\alpha^2)^{-1}, \quad c_\alpha^2 = \Lambda_{0\alpha}\rho_0^{-1} \end{aligned} \tag{1.12}$$

Here $j_n(q, r)$ is the spherical Bessel function.
We represent $j_n(qr)$ in the form

$$j_n(qr) = \sum_{k=0}^\infty \frac{(-1)^k q^{n+2k} r^{n+2k}}{2^k\, k!\, (2n + 2k + 1)!!}$$

We then obtain

$$P^p_{n(\alpha)} = \sum_{k=0}^\infty P^{pk}_{n(\alpha)} q^{2k}, \quad P^{pk}_{n(\alpha)} = \frac{(-1)^k k_\alpha^{-p} q^n}{(2n + 2k + 1)!!\, k!\, 2^k} \times \int_0^\infty R(r) \exp(ik_\alpha r)\, r^{n+2k-p+2}\, dr \tag{1.13}$$

Taking (1.13) into account, we write (1.12) in the form

$$\begin{aligned} &\gamma_i^*(q) = \sum_{k=0}^\infty \gamma_i^{(k)} q^{2k}, \quad i = 1, 2 \\ &\gamma_1^{(k)} = -\theta_t [P^{2(k+2)}_{2(t)} - iP^{2(k+2)}_{2(t)} - P^{2k}_{1(t)}\varkappa_t] + 2^{-1}\gamma_2^{(k)} \\ &\gamma_2^{(k)} = -2\theta_\alpha [P^{3(k+4)}_{4(\alpha)} - 3iP^{4(k+4)}_{4(\alpha)} + 3P^{5(k+4)}_{4(\alpha)} + 2P^{4(k+2)}_{3(\alpha)}\varkappa_\alpha + 2iP^{3(k+2)}_{3(\alpha)}\varkappa_\alpha + P^{3k}_{2(\alpha)}]^l_t \end{aligned} \tag{1.14}$$

Therefore, the eigenvalues of the polarizability operator $\Gamma^*(q, \omega)$ are entire functions of q^2 in the complex q^2 plane.

2. We now examine the standard problem of harmonic wave propagation in a random inhomogeneous medium. The simplest three-dimensional problem is described by the Helmholtz equation /4-6/, for which it is possible to obtain analytic expressions in transparent form.

This is because the dispersion equation has a simple form, and the roots can be evaluated fairly easily. Correspondingly, the frequency dependences of the velocities and the scattering coefficient have a comparatively simple form. The Helmholtz equation is used as a standard for investigating wave propagation in inhomogeneous media in electrodynamics /5/, and in elasticity theory /2,4/. We ignore effects associated with wave polarization /5/ here. It is established that for short (high-frequency) waves the dynamic equations for inhomogeneous media can be split into Helmholtz equations for longitudinal and transverse waves /2/.

We will show that the eigenvalues of an elastic operator for a stochastically inhomogeneous elastic medium are expressed in terms of the eigenvalues of a generalized Helmholtz operator while the roots of the dispersion equations (1.7) and the eigenvectors (1.5) are expressed in terms of the roots and eigenvectors of the corresponding standard problem. We write the standard equation in the form

$$\Delta\varphi + k^2\lambda(\mathbf{x})\,\varphi = 0 \tag{2.1}$$

where k^2 is the square of the wave number of the effective homogeneous medium, $\lambda(\mathbf{x}) = n^2(\mathbf{x})$ is the square of the refractive index $n(\mathbf{x}) = c_0 c^{-1}(\mathbf{x})$ which is a random function of the space coordinates, $c(\mathbf{x})$ is the velocity in the inhomogeneous medium, and c_0 is the velocity in the effective homogeneous medium.

The averaged equation (2.1) is later considered as standard for operator equations corresponding to (1.7), which the longitudinal and transverse waves $\langle u^\alpha \rangle$ of the elastic problem satisfy, on the average. We consequently ascribe the index α to the quantities $\varphi, k, \lambda, c(\mathbf{x}), c_0$, where $\alpha = l$ when considering the longitudinal waves and $\alpha = t$ when considering the transverse waves on the average. Henceforth, the index α will not be written on the quantities mentioned. We introduce the effective operator of the standard problem by the relationship

$$\langle \lambda\varphi \rangle = \Lambda^* \langle \varphi \rangle \tag{2.2}$$

Taking account of (2.2) when averaging (2.1), we obtain standard equation for $\langle \varphi \rangle$

$$\Delta \langle \varphi \rangle + k^2 \Lambda^* \langle \varphi \rangle = 0 \tag{2.3}$$

For a statistically isotropic homogeneous medium Λ^* has the form (1.4), while the eigenvectors have the form (1.5) although $\lambda^*(\mathbf{x}-\mathbf{x}_1)$ and $u(\mathbf{q})$ are different.

We write the dispersion equation in the form

$$\Delta = q^2 - k^2\Lambda^*(q, \omega) = 0, \quad \Lambda^* = \lambda_0 (1 - \Gamma^*(q, \omega))^{-1} \tag{2.4}$$

$$F^*(\mathbf{q}) = \int f^*(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}, \quad (F^* = \Lambda^*, \Gamma^*; \; f^* = \lambda^*, \gamma^*)$$

For an arbitrary correlation function $R(r) = \langle \gamma(\mathbf{x}) \gamma(\mathbf{x}_1) \rangle$ we have

$$\Gamma^*(q, \omega) = ak^2 \int_0^\infty R(r) e^{ikr} j_0(qr)\, dr = akP_0^1(q, \omega) = \tag{2.5}$$

$$a\varkappa q^{-1} Q_0^1, \quad Q_n^p = \int_0^\infty j_n(z) z^{2-p} e^{i\varkappa z} R(z\alpha^{-1})\, dz$$

$$\varkappa = kq^{-1}, \quad \alpha = aq, \quad P_n^p = \varkappa^2 q^{-3} Q_n^p$$

Using the recurrence relations for spherical Bessel functions /10/, we obtain a recurrence formula for Q_n^p

$$Q_{n+1}^p = (2n+1) Q_n^{p+1} - Q_{n-1}^p \tag{2.6}$$

We use the notation

$$2 - p = m - n, \quad Q_n^m = \int_0^\infty j_n(z) z^{m-n} e^{i\varkappa z} R(z\alpha^{-1})\, dz$$

We then obtain

$$Q_{n+1}^{m+1} = (2n+1) Q_n^{m-1} - Q_{n-1}^{m-1} \tag{2.7}$$

Formulas (2.6) and (2.7) do not provide any possibility of expressing the eigenvalues $\Gamma^*(q, \omega)$ of the elastic problem for an arbitrary correlation function in terms of the $\Gamma^*(q, \omega)$ of the corresponding standard problem.

We will examine the class of media described by correlation functions of the form

$$R(r) = A (ra^{-1})^k e^{-dr}, \quad k = -1, 0 \tag{2.8}$$

where a is the correlation radius, and A, d are complex quantities.

We set $k = 0, A = R_0, d = a^{-1}$; we then obtain in (2.5)

$$\Gamma^*(q, \omega) = k^2 s^{-2} Q_0^1, \quad Q_0^1 = F\left(1, \frac{3}{2}; \frac{3}{2}; -\beta^{-2}\right) = (1 + \beta^{-2})^{-1}, \quad \beta = sq^{-1}, \quad s = a^{-1} - ik \tag{2.9}$$

Setting $k = -1, A = R_0, d = a^{-1}$ in (2.8), we obtain

$$\Gamma^*(q, \omega) = R_0 ak^2 \beta^{-1} F\left(\frac{1}{2}, 1; \frac{3}{2}; -\beta^{-2}\right) = R_0 ak^2 Q_0^0(\beta) = R_0 ak^2 \operatorname{arctg} \beta^{-1} \tag{2.10}$$

The correlation function of a Markov field of the form

$$R(r) = R_0 j_0(ra^{-1}) = \frac{aR_0}{2i} e^{i(ra^{-1})(-1)^n} \Big|_1^2 \tag{2.11}$$

$$A = aR_0 (2i)^{-1}, \; d_n = -(-1)^n ia, \; k = -1, \; f_n|_1^2 = f_2 - f_1$$

$$\Gamma^*(q, \omega) = 2[Q_0^0(\beta_1^{-1}) + Q_0^0(\beta_2^{-1})], \quad \beta_1 = (1 - ak)\alpha^{-1}$$

$$\beta_2 = (1 + ak)\alpha^{-1}, \quad Q_0^0 = \frac{1}{2i} \ln \frac{\beta + 1}{\beta - 1} = \frac{1}{i\beta} F\left(\frac{1}{2}, 1; \frac{3}{2}; \beta^{-2}\right)$$

can be reduced to the form (2.8),

For the kinds of correlation functions under consideration, the coefficients $P_n^p(Q_n^m)$, and therefore also $\Gamma^*(q, \omega)$ in (2.9)-(2.11), are evaluated in terms of the hypergeometric functions

$$Q_n^m = \frac{m!\, \beta^{m-1}}{(2n+1)!!} F\left(\frac{m+1}{2}, \frac{m+2}{2}; \frac{2n+3}{2}; -\beta^{-2}\right)$$

Using the well-known Gauss recursion formulas for the hypergeometric functions /9,10/, in addition to (2.6) and (2.7), recurrence formulas can be obtained for $Q_n^m\,(P_n^p)$ for correlation functions of the form

$$Q_{n+1}^m = \frac{(2n+1)\beta^2 + (4n-2m+1)}{(2n+1-m)(2n+2-m)}\, Q_n{}^m - \frac{(1+\beta^2)\, Q_{n-1}^m}{(2n+1-m)(2n+2-m)} \tag{2.12}$$
$$Q_n^{m+1} = \beta^{-1}\,[Q_{n-1}^m - (2n-m)\, Q_n{}^m]$$

We consider the calculation of the eigenvalues of the polarizability operator $\Gamma^*(q,\omega)$ using formulas (1.11) and (1.12). Formulas (2.6),(2.7) and (2.12) enable us to express $Q_n{}^m$ in terms of $Q_1{}^1$ and $Q_0{}^1$

$$Q_n{}^m = A_n{}^m Q_0{}^1 + B_n{}^m Q_1{}^1 \tag{2.13}$$

According to (2.13), we obtain the expression for $\Gamma^*(q,\omega)$ in the form

$$\Gamma^* = \tau_1^{(0)} Q_1{}^1 + \tau_2^{(0)} Q_0{}^1, \quad \Gamma_2{}^* = \tau_1^{(2)} Q_1{}^1 + \tau_2^{(2)} Q_0{}^1 \tag{2.14}$$
$$\tau_1^{(0)} = -3q^{-3}(3\theta_t D_1 + 11\theta_\alpha D_2), \quad \tau_2^{(0)} = -3q^{-3}(3\theta_t C_1 + 11\theta_\alpha C_2)$$
$$\tau_1^{(2)} = -6q^{-3}(\theta_t D_1 + 2\theta_\alpha D_2), \quad \tau_2^{(2)} = -6q^{-3}(\theta_t C_1 + 2\theta_\alpha C_2)$$
$$C_1 = \varkappa_t^{-3} A_2{}^1 - 3i\varkappa_t^{-2} A_1{}^0 + i\varkappa_t^{-2} A_0{}^0$$
$$D_1 = \varkappa_t^{-3} B_2{}^1 - 3i\varkappa_t^{-2} B_1{}^0 + i\varkappa_t^{-2} B_0{}^0$$
$$C_2 = 3\varkappa_\alpha^{-5} A_4{}^1 - 3i\varkappa_\alpha^{-4} A_4{}^2 + \varkappa_\alpha^{-1}(1-\varkappa_\alpha^{-2})\, A_2{}^1 + 9\varkappa_\alpha^{-3} A_3{}^1 + 8i\varkappa_\alpha^{-2} A_1{}^0$$
$$D_2 = 3\varkappa_\alpha^{-5} B_4{}^1 - 3i\varkappa_\alpha^{-4} B_4{}^2 + \varkappa_\alpha(1-\varkappa_\alpha^{-2})\, B_2{}^1 + 9\varkappa_\alpha^{-3} B_3{}^1 + 8i\varkappa_\alpha^{-2} B_1{}^0$$
$$A_4{}^1 = -\frac{b\,(35\beta^4 + 79\beta^2 + 57)}{7!}, \quad B_4{}^1 = \frac{b^3}{6!!}$$
$$A_4{}^2 = \frac{b\,(35\beta^4 + 52\beta^2 + 15)}{7!\,\beta}, \quad B_4{}^2 = -\frac{b\,(1+3\beta^2)}{6!!\,\beta}$$
$$A_3{}^1 = -\frac{b\,(5\beta^2+7)}{5!}, \quad B_3{}^1 = \frac{b^2}{4!!}, \quad A_2{}^1 = -\frac{b}{3!}, \quad B_2{}^1 = \frac{b}{2!}$$
$$A_1{}^0 = \frac{b}{2\beta}, \quad B_1{}^0 = -A_1{}^0, \quad A_0{}^0 = b\beta^{-1}, \quad B_0{}^0 = -\beta^{-1}, \quad b = 1+\beta^2$$

Taking account of (2.9),(2.10), we obtain the following representation from (2.14)

$$\Gamma^* = \tau_1^{(0)} + T_1^{(0)}\Gamma^{(1)} + T_2^{(0)}\Gamma^{(2)}, \quad \Gamma_2{}^* = \tau_1^{(2)} + T_1^{(2)}\Gamma^{(1)} + T_2^{(2)}\Gamma^{(2)} \tag{2.15}$$
$$\Gamma^{(1)} = ak^2 R_0 Q_0{}^0, \quad \Gamma^{(2)} = k^2 s^{-2} R_0 Q_0{}^1, \quad Q_1{}^1 = 1 - \beta Q_0{}^0$$
$$T_1^{(0)} = -\beta\,(R_0 a k^2)^{-1}\,\tau_1^{(0)}, \quad T_2^{(0)} = R_0^{-1} k^{-2} s^2 \tau_2^{(2)}$$
$$T_1^{(2)} = -\beta\,(R_0 a k^2)^{-1}\,\tau_1^{(2)}, \quad T_2^{(2)} = R_0^{-1} k^{-2} s^2 \tau_2^{(2)}$$

Therefore, the eigenvalues of the polarizability operator of the elastic problem $\Gamma^*(q,\omega), \Gamma_2{}^*(q,\omega)$ are expressed in terms of the eigenvalues $\Gamma^{(2)}(q,\omega), \Gamma^{(1)}(q,\omega)$ of the standard problems (2.9),(2.10), respectively. Because of (1.9), the eigenvalues of the elastic operator $\Lambda_\alpha(q_\alpha,\omega)$ are expressed in terms of $\Gamma^{(i)}(q_\alpha,\omega)$ of two standard problems. The eigenvalues Γ^* in the standard problem $\Gamma^{(i)}(q,\omega)$ are the polarizability operator, taking account of multiple scattering, and are related to the eigenvalues $\Lambda^{(i)}(q,\omega)$ of the operator Λ^* by the formula

$$\Gamma^*(q,\omega) = (\Lambda^*(q,\omega) - \lambda_0)\,\Lambda^{*-1}(q,\omega) \tag{2.16}$$

Thus, the eigenvalues of the elastic operator are expressed in terms of the eigenvalues of the operators Λ^* of the corresponding standard problems. For example, for an elastic medium with the exponential correlation function (2.8) with $k=0,\ A=R_0,\ d=a^{-1}$, the eigenvalues $\Gamma^*(q,\omega)$ are expressed by means of (2.15) in terms of the eigenvalue $\Gamma^{(1)}(q,\omega)$ of the standard problem for a medium with the same correlation function and in terms of the eigenvalue $\Gamma^{(2)}(q,\omega)$ of the standard problem for a medium with the correlation function (2.8) with $k=-1,\ A=R_0,\ d=a^{-1}$.

3. We examine evaluation of the eigenvectors and roots of dispersion equations of the elastic problem in terms of the corresponding quantities of the standard problem. On average, the longitudinal and transverse waves in an elastic medium satisfy the equations

$$\Delta\Lambda_{\alpha u}^{*-1}\langle u^\alpha\rangle + k_\alpha{}^2\langle u^\alpha\rangle = 0,\ \alpha = l,\ t \tag{3.1}$$

The corresponding standard equations have the form

$$\Delta\Lambda_{\alpha\varphi}^{*-1}\langle \varphi^\alpha\rangle + k_\alpha{}^2\langle \varphi^\alpha\rangle = 0,\ \alpha = l,\ t \tag{3.2}$$

Here $\Lambda_{\alpha\varphi}^{*-1}$ is the inverse operator to Λ^* in (2.2). We later omit the subscript α but retaining the subscript u to denote quantities in the elastic problem and the subscript φ for the standard problem.

The integral equation

$$\langle u^\alpha\rangle = \langle\varphi^\alpha\rangle - \int G_{\alpha\varphi}(\mathbf{x}-\mathbf{x}_1)\Delta\Lambda_\alpha'(\mathbf{x}_1)\langle u^\alpha\rangle d\mathbf{x}_1 \tag{3.3}$$
$$\Lambda_\alpha' = \Lambda_{\alpha u}^* - \Lambda_{\alpha\varphi}^*, \quad (\Delta\Lambda_{\alpha\varphi}^* + k^2)G_{\alpha\varphi} = \delta(\mathbf{x}-\mathbf{x}_1)$$

is equivalent to (3.1) and (3.2).

Solving (3.3) by successive iterations, we obtain

$$\langle u^\alpha\rangle = T_\alpha^*\langle\varphi^\alpha\rangle, \quad T_\alpha^* = 1 + \int G_{\alpha\varphi}\Delta\Lambda_\alpha' d\mathbf{x}_1 + \iint G_{\alpha\varphi}G_{\alpha\varphi}\Delta\Lambda_\alpha'\Delta\Lambda_\alpha' d\mathbf{x}_1 d\mathbf{x}_2 + \ldots \tag{3.4}$$

There is no summation over repeated subscripts in (3.1)-(3.4). The solution $\langle u^\alpha\rangle$ in the form of the scattering series (3.4) takes account of the successive rescattering of the "standard" wave by the average inhomogeneities of the elastic medium.

The dispersion equations corresponding to (3.1) and (3.2) are written, respectively, in the form

$$q_{\alpha\beta}^2 = k_\alpha^2\Lambda_{\alpha\beta}^{*-1}(q_{\alpha\beta}, \omega), \; \beta = u, \varphi \tag{3.5}$$

From (3.5) we obtain an equation to calculate $q_{\alpha u}^2$ in terms of $q_{\alpha\varphi}^2$

$$q_{\alpha u}^2 = q_{\alpha\varphi}^2 - k_\alpha^2\Lambda_{\alpha u\varphi}'(q_{\alpha u}, q_{\alpha\varphi}, \omega), \quad \Lambda_{\alpha u\varphi}' = \Lambda_{\alpha u}^{*-1} - \Lambda_{\alpha\varphi}^{*-1} \tag{3.6}$$

Solving (3.6) by successive iterations, we find

$$q_{u(0)} = q_\varphi, \; n = 1, 2, \ldots \tag{3.7}$$
$$q_{u(n)}^2 = q_\varphi^2 - k^2\Lambda_{u\varphi}'(q_{u(n-1)}, q_\varphi, \omega)$$

The eigenvectors of the operators Λ_u^{*-1} and Λ_φ^{*-1} for a statistically isotropic homogeneous medium have the form $\exp(i\mathbf{q}_u\cdot\mathbf{x})$ and $\exp(i\mathbf{q}_\varphi\cdot\mathbf{x})$. The eigenfunctions of an elastic operator are expressed in terms of the eigenfunctions of the standard operator

$$\exp(i\mathbf{q}_u\cdot\mathbf{x}) = \exp(i\mathbf{q}_\varphi\cdot\mathbf{x})\exp(i\boldsymbol{\psi}\cdot\mathbf{x}) \tag{3.8}$$
$$\boldsymbol{\psi} = \mathbf{q}_\varphi k^2\Lambda_{u\varphi}'(\mathbf{q}_\varphi, \omega) + q_u\cdot\mathbf{q}'$$
$$\mathbf{q}' = \mathbf{q}_u^0 - \mathbf{q}_\varphi^0, \; \mathbf{q}_\beta^0 = \mathbf{q}_\beta q_\beta^{-1}, \quad \beta = u, \varphi$$

The first term in the vector $\boldsymbol{\psi}$ in (3.8) takes account of the difference of $\mathbf{q}_u$ from $\mathbf{q}_\varphi$ because of the difference in the eigenvalues $\Lambda_{u\varphi}'$, while the second is because of the difference in the directions of the elastic and standard vectors.

Calculation of the approximate eigenvectors and the roots of the elastic problem in terms of the standard values is a result of extending the standard operator to the elastic operator /11,12/. Note that (3.7) enables us to relate the distribution of the roots /13/ of the elastic and standard problems.

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APPLICATION OF DUALITY METHODS IN PROBLEMS OF OPTIMIZING THE SHAPE OF ELASTIC BODIES*

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A method is proposed for obtaining estimates of the magnitude of the global extremum in plate and three-dimensional body shape-optimization problems. This enables an estimate to be made of the ultimate possibilities of optimization. In certain cases, a control is constructed successfully for which the values of the objective functional will be close, and sometimes even equal, to the magnitude of the global extremum.

1. **Free vibrations of thin plates.** Let there be a domain $\Omega \in R^2$ with piecewise-smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. The frequency ω of free vibrations of a plate of thickness h is given by the following relations:

$$\omega^2 = \min_{w \in V} \Phi(h, w); \quad \Phi(h, w) = \Pi(h, w)/T(h, w) \tag{1.1}$$

$$\Pi(h, w) = \int_\Omega Dh^3\psi(x, y)\, d\Omega; \quad T(h, w) = \int_\Omega \rho h \varphi(x, y)\, d\Omega$$

$$D = E/(12(1-\nu^2)), \quad \varphi = w^2, \quad \psi = (\Delta w)^2 - 2(1-\nu)\cdot$$

$$\cdot\, (w_{,xx} w_{,yy} - w^2_{,xy})$$

$$V = \{v \mid v \in W_2^2(\Omega); \quad v = v_{,n} = 0 \text{ on } \Gamma_1, \quad v = 0 \text{ on } \Gamma_2\}$$

Here E is Young's modulus, ν is Poisson's ratio, ρ is the density, w is the deflection, x, y are Cartesian coordinates of a point, and $v_{,n}$ denotes the derivative along the normal to the contour Γ. The optimization problem is as follows: it is required to find h^* and w^* such that

$$\Phi(h^*, w^*) = \sup_{h \in H} \inf_{w \in V} \Phi(h, w) \tag{1.2}$$

$$H = \left\{h \in L_\infty(\Omega) \,\middle|\, \int_\Omega h\, d\Omega = h_3 \operatorname{mes}\Omega, \ h_1 \leqslant h \leqslant h_2\right\}$$

$$h_2 > h_3 > h_1 > 0$$

where $\operatorname{mes}\Omega$ denotes the Lebesgue measure of the domain Ω.

It is known that in problems of this kind, the existence of generalized solutions /5,6/ is possible in addition to piecewise-smooth solutions /1-4/. Moreover, problem (1.2) is non-convex; consequently, different numerical algorithms only result in locally optimal solutions /7,8/. However, attempts can be made to find the function $h \in H$ for which the value of the objective functional is less than the supremum by a certain small quantity ε. For this it is necessary to estimate the value of the supremum, as can be done by using the dual problem.

The following problem is called the dual of the original /9/: Find h^*, w^* such that

$$\Phi(h^*, w^*) = \inf_{w \in V} \sup_{h \in H} \Phi(h, w) \tag{1.3}$$

The following inequalities are obviously valid

$$\sup_{w \in V} \inf_{h \in H} \Phi(h, w) \leqslant \inf_{h \in H} \sup_{w \in V} \Phi(h, w) \tag{1.4}$$

and can be used to construct upper bounds for the magnitude of the supremum in problem (1.2). We use the notation

$$Q_0 = \sup_{h \in H} \Phi(h, w_0), \quad w_0 \in V \tag{1.5}$$

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